The incommensurate phase of a bilayer quantum Hall state is found to have a “rippled” dipole charge density whenever the layers are unbalanced. This tunable dipole-density-wave instability could be detected by sensitive capacitance measurements and by anisotropic transport. We demonstrate this explicitly by carrying out a Hartree-Fock calculation of the layer densities and capacitance for a double-layer quantum Hall state at a total filling factor of 1.

I. INTRODUCTION

The combination of reduced dimensionality and strong interparticle interactions can have spectacular effects on the nature of the ground-state and dynamical properties of many-particle systems. This is especially evident in the fractional quantum Hall regime,\(^1\) where a strong magnetic field is applied perpendicular to a two-dimensional electron gas at very low temperatures. The powerful magnetic field quenches the kinetic energy of the electrons, so that interactions between electrons dominate the energetics. The result is a highly correlated, incompressible quantum-liquid ground state that supports fractionally charged excitations.\(^3\) Even at integer filling factors (especially \(\nu_T = 1\)), the combination of the quenched kinetic energy of interacting electrons plus extra electronic degrees of freedom (i.e., spin or multiple layers) can give rise to correlated ground states that support remarkable topological excitations, such as charged skyrmions.\(^4,5\)

Another important class of systems in which reduced dimensionality and interactions strongly affect the ground state and dynamics are systems with charge-density-wave or spin-density-wave ground states,\(^6\) and also systems that exhibit commensurate-incommensurate (CI) transitions.\(^7,8\) Systems with charge- or spin-density-wave ground states occur most famously (although not exclusively) in quasi-one-dimensional materials, where the reduced dimensionality enhances fluctuation effects and leads to broken-symmetry ground states.\(^9\) Systems exhibiting CI transitions have broken translational symmetry states with rich structures, most notably arrays of domain walls that arise from the competition between interparticle interactions and external periodic potentials.\(^7,8\)

Double-layer quantum Hall (2LQH) systems can show both types of behavior: they support unusual fractionally charged topological excitations,\(^9,10\) and they apparently exhibit a CI transition to a state with broken translational symmetry in the presence of a sufficiently strong in-plane magnetic field.\(^8\) It is interesting that in 2LQH systems, broken translational symmetry can coexist with the hidden off-diagonal long-range order characteristic of quantum Hall states. The evidence for a CI transition in 2LQH systems is, however, indirect. Activation energy measurements show that a sufficiently large in-plane magnetic field drives a transition between two different types of many-body ground states,\(^12\) and the strength of the in-plane magnetic field and the size of the energy gaps are consistent with the CI scenario.\(^9,12\) In order to establish the nature of the competing quantum Hall ground states and the transition between them, additional types of experimental measurements are needed. Various types of experimental signatures of the CI scenario have been proposed, including predictions of the form of the energy gap,\(^13\) a predicted Kosterlitz-Thouless transition,\(^10\) anisotropic transport in narrow samples,\(^14\) and the field dependence of the in-plane magnetization.\(^15,16\) Here we propose that the CI transition in 2LQH systems could be studied directly by capacitance measurements, and by anisotropic transport produced by the dipole-density wave described below.

This paper examines a novel effect of unequal layer densities on the CI transition in 2LQH systems. (Interestingly, unequal layer densities can actually enhance the stability of the 2LQH state at total filling factor \(\nu_T = 1\).\(^17,18\)) It is found that when the layer densities are not equal, they go from being uniform in the commensurate phase to becoming “rippled” in the incommensurate phase; this allows the CI transition to be detected by sensitive capacitance measurements.\(^17\) The layer imbalance can be produced in two ways: most commonly by an external bias, but perhaps also spontaneously in a tilted sample with an unusually small capacitive charging energy and a sufficiently large interlayer tunneling.\(^19,20\) We will focus here on bias-driven imbalance, since it is easier to achieve experimentally. In addition, capacitive techniques provide a quantitative measure of the interlayer exchange and pseudospin stiffness in 2LQH systems. This is illustrated in the following sections by a Hartree-Fock calculation of the layer densities and capacitances for a 2LQH state at a total filling factor \(\nu_T = 1\). Similar effects should, in principle, occur at other filling factors, although \(\nu_T = 1\) is probably most promising for experimental observation.

II. RIPPLED STATE

Interlayer Coulomb interactions at low filling factors can stabilize 2LQH states when the layer spacing is comparable to the separation between electrons within the layers.\(^2\) Even at a total “integer” filling factor \(\nu_T = 1\), experiments indicate that the quantum-Hall ground states are stabilized by Coulomb interactions and do not require interlayer tunneling for...
their existence. Further evidence of the rich variety of 2LQH states at $n_\parallel=1$ comes from measuring the effects of an in-plane magnetic field, which induces a transition between two types of quantum Hall ground states. These effects have been discussed in terms of an unusual broken-symmetry quantum Hall ground states that exhibit spontaneous interlayer (phase) coherence (SILC).

At sufficiently small layer separation, a 2LQH system is an unusual quantum itinerant ferromagnet. The 2LQH quantum ferromagnet exhibits a rich variety of ground states, phase transitions, and charged quantum Hall states. Their existence has been explained theoretically and experimentally, and it was found that application of a sufficiently strong parallel magnetic field between two types of quantum Hall ground states.

A. Effective Hamiltonian

Formally, it is simplest to obtain the ground-state characteristics of the SILC 2LQH state from the energy per unit area within a gradient approximation in which the pseudospin $\mathbf{m}(\mathbf{r})$ is assumed to vary slowly on the scale of the magnetic length $l$. In doing so, it is convenient to specify the order parameter $\mathbf{m}(\mathbf{r})$ in terms of two quantities: $m_z(\mathbf{r})$, the local difference in layer occupancies, and $\theta(\mathbf{r})$, the projected angle of $\mathbf{m}$ in the $xy$ plane measured with respect to the $x$ axis. For constant $m_z = n_1-n_2$ and in-plane magnetic field $B_\parallel$, the energy per unit area of the SILC 2LQH state has the form

$$\mathcal{E} = \frac{1}{2\pi l^2} \left[ -t \cos \tilde{\theta} + \frac{1}{2} \rho_s l^2 (\nabla \tilde{\theta} - \mathbf{Q})^2 \right]$$

$$+ \frac{U}{4} m_z^2 - \frac{1}{2} V_s m_z^2, \quad (2.1)$$

where $\mathcal{E}$ has been expressed in terms of $\tilde{\theta} = \theta + \mathbf{Q} \cdot \mathbf{r}$, $\mathbf{Q} = \mathbf{z} \times \mathbf{B}_\parallel l / \phi_0$, $d$ is the interlayer spacing, and $\phi_0 = h/e$ is the magnetic-flux quantum. Mean-field equations for $\tilde{\theta}$ and $m_z$ are obtained by minimizing Eq. (2.1) with respect to those same quantities.

The first two terms of Eq. (2.1) constitute the Pokrovsky-Talapov (PT) model, with coefficients that, in the Hartree-Fock approximation (HFA), depend on $m_z$ according to

$$t = t_0 \sqrt{1-m_z^2} e^{-Q^2/4} = t_0 \sqrt{4n_1n_2} e^{-Q^2/4}$$

$$\rho_s = \rho_0 (1-m_z^2) = 4n_1n_2\rho_0, \quad (2.2)$$

where $t_0$ is the tunneling-matrix element when $n_1=n_2$; it is equal to half the symmetric-antisymmetric gap $\Delta_{\pm.\pm}$. In the presence of a parallel magnetic field,

$$t_0 \bigg|_{t_0} \exp \left( -Q^2/4 \right), \quad (2.3)$$

which is a single-body effect. The interlayer pseudospin stiffness when the layers are balanced is

$$\rho_0 = \frac{e^2}{4\pi e} \frac{1}{16\pi} \int_0^\infty dxx^2 e^{-x^2/\rho^2} e^{-xdd/\ell}, \quad (2.4)$$

in the HFA. The value of $\rho_0$ will be reduced due to quantum fluctuations and finite-thickness effects. By adjusting the front and back gate voltages of the sample, $n_1$ and $n_2$ may be varied (with $n_1+n_2=1$), thereby allowing $t$ and $\rho_s$ to be adjusted.

The third term (quadratic in $m_z$) in the energy density [Eq. (2.1)] is a capacitive charging energy that favors equal layer densities. The capacitive energy $U$ is given in terms of the electrostatic Hartree energy ($D_1$) and the intralayer ($E_0$) and interlayer ($\bar{E}_1$) exchange energies by

$$U = D_1 - E_0 + \bar{E}_1,$$

$$D_1 = \frac{e^2}{4\pi e} dl,$$

$$\bar{E}_1 = \frac{e^2}{4\pi e} I_1 = \frac{e^2}{4\pi e} \int_0^\infty dxe^{-x^2/\rho^2} e^{-xd/\ell}, \quad (2.5)$$

where the exchange integrals $I_1$ have been evaluated in the HFA. Most treatments of the SILC 2LQH state have been for equal layer densities ($m_z=0$), since there is a significant cost in capacitive charging energy to unbalance the layers. However, an application of back and front gate voltages allows charge to be transferred from one layer to another, giving rise to a tunable nonzero value for $m_z$ in the 2LQH ground state. The effects of charge-transfer imbalance were studied both theoretically and experimentally, and it was found that charge imbalance can actually increase the stability of the 2LQH state.

We shall estimate the numerical values of our results for a hypothetical “typical” GaAs ($m^* = 0.07m_e$, $\epsilon_{\parallel} = 13$) 2LQH sample, with a total density $n_f = 1.0 \times 10^{11}$ cm$^{-2}$, a layer (midwell to midwell) separation $d = 20$ nm, and a tunneling energy $t_0 = 0.5$ meV ($\Delta_{\parallel.\parallel} = 11.6$ K). Such a sample would have $l = 12.6$ nm, $d/l = 1.6$, $\phi_0 = 6.9$ meV for $n_f = 1$, and $\epsilon^2/4\pi e \ell = 8.8$ meV. In the HFA, $\rho_0 \approx 0.03$ meV, $D_1 = 14$ meV, and $U = 7.4$ meV.

The effect of the gate voltages is described by the last ($V_s$) term in the energy density in terms of effective filling factors $\tilde{v}_f$ and $\tilde{v}_b$ for the front (F) and back (B) gates:

$$V_s = (\tilde{v}_f - \tilde{v}_b) D_1. \quad (2.6)$$

The effective filling factors $\tilde{v}_\alpha$ (where $\alpha = F,B$) are defined by the the electric fields $E_\alpha$ produced by the front and back
gates through Gauss’ law $E(y) = e \nu_d / 2\pi l^2 e$, where $e$ (approximately $13e_0$ for GaAs) is the dielectric constant appropriate to the 2LQH sample.

B. Parallel magnetic field

When $Q \neq 0$, the pseudospin stiffness $\rho_z$ competes with the rotating Zeeman pseudofield $\nu \cos \tilde{\theta}$ to determine the spatial orientations of the pseudospins. Minimizing $\xi$ in Eq. (2.1) with respect to $\tilde{\theta}$ gives the two-dimensional sine-Gordon equation

$$\xi^2 \nabla^2 \tilde{\theta} = \sin \tilde{\theta},$$

(2.7)

where the width of the soliton is proportional to the length,

$$\xi = \sqrt{2 \pi \rho_s / l} = \xi_0 (1 - m_z^2)^{1/4},$$

(2.8)

which for our hypothetical sample gives $\xi_0 = 17$ nm. The soliton width $\xi$ sets the scale for spatial variations of the pseudospin $\mathbf{m}(\mathbf{r})$; thus the condition for the validity of the gradient approximation is that $\xi$ be significantly larger than the magnetic length $l$.

For sufficiently small $Q$, Eq. (2.1) is minimized by $\tilde{\theta}(\mathbf{r}) = 0$. This is the commensurate (C) phase, and in this phase the pseudospins align themselves with the rotating Zeeman pseudofield, so that $\tilde{\theta} = -Q \cdot \mathbf{r}$. However, above a critical value of $B_1$ corresponding to $Q_c = (4/\pi) \xi_0 (1/\sqrt{l^2 \pi \rho_z})$, it becomes energetically favorable to produce dislocation lines (solitons). The soliton widths are of order $\xi$. Solitons proliferate rapidly for $Q > Q_c$ (incommensurate phase) because they repel each other only very weakly (at zero temperature). The resulting array of solitons breaks the translational symmetry of the 2LQH ground state by forming a SL. For large $Q > Q_c$, the rapidly varying tunneling phase factor causes the pseudospins to behave (nearly) as if $r = 0$. In the HFA, the critical value of the in-plane field varies with the layer filling factors like $Q_c \approx (1 - m_z^2)^{-1/4}$, thus tuning the layer filling factors ($m_z$) via gate voltages allows the location ($Q_c$) of the CI transition to be fine tuned.

The density of soliton lines in the incommensurate phase is proportional to the soliton wave vector $Q_s = 2\pi / L_s$, where $L_s$ is the spacing between solitons in the SL. The soliton density (proportional to $Q_s$) is calculated as a function of the in-plane field (proportional to $Q$) via two equations involving an intermediate parameter $\eta$. The parameter $\eta$ is defined by

$$Q_s / Q_c = (\pi / 2)^2 \cdot \left[ \eta K(\eta) \right],$$

(2.9)

approaches 1 near the CI transition ($Q_s \rightarrow 0$), and goes to 0 deep in the incommensurate phase ($Q_s \rightarrow \infty$). Here $K(\eta)$ is the complete elliptic integral of the first kind.$^{31}$

When the layers are balanced ($\nu_1 = \nu_2$), minimizing the total energy with respect to $Q_s$ gives

$$Q / Q_c = E(\eta) / \eta,$$

(2.10)

where $E(\eta)$ is the complete elliptic integral of the second kind.$^{32}$ Equations (2.10) and (2.9) together determine the soliton density ($Q_s$) as a function of the in-plane magnetic field (or $Q$). $Q_s / Q_c$ is plotted as a function of $Q / Q_c$ in Fig. 1, for the balanced case$^{16}$ ($m_z = 0$, solid curve) and for the rippled state ($m_z = 0.5$, dashed curve). Note the abrupt rise in soliton density for $Q > Q_c$. The compressional stiffness $K_1$ of the SL$^{16}$ is proportional to the slope of the $Q_s$ versus $Q$ curves in Fig. 1 (see Ref. 16):

$$\rho_z / K_1 = \partial Q_s / \partial Q_c.$$  

(2.11)

As expected, $K_1$ vanishes as $Q \rightarrow Q_c$.

C. Rippled layer imbalance

Minimizing Eq. (2.1) with respect to variations in $m_z$ gives

$$m_z = V_d / \left[ 2t \cos \tilde{\theta} - 4 \pi l^2 \rho_z (\nabla \tilde{\theta} - Q_c)^2 \right] (1 - m_z^2).$$

(2.12)

This equation determines (in the Hartree-Fock gradient approximation) the filling factor of each layer for the $\nu_T = 1$ SILC state. For sufficiently small in-plane magnetic fields ($Q < Q_c$), $\tilde{\theta} = 0$ (the commensurate state), and the assumption that $m_z$ is constant is self-consistent, provided that $t \ll U$. However, when $Q \approx Q_c$, the quantum Hall ground state breaks translational invariance: $\tilde{\theta}$ is not spatially uniform and the soliton-lattice state is obtained. When $Q \approx Q_c$ and $\tilde{\nu}_1 \neq \tilde{\nu}_2$, Eq. (2.12) shows that $m_z$ also breaks translation invariance, and one obtains “rippled” layer densities. Thus uniform $m_z$ is not consistent with the broken translation symmetry of $\tilde{\theta} = 0$ in the incommensurate phase, when $\tilde{\nu}_1 \neq \tilde{\nu}_2$. The resulting behavior of $m_z(\mathbf{r})$ is illustrated in Fig. 2.

It is important to note that Eq. (2.12) is valid even in the incommensurate state of spatially varying $\tilde{\theta}$, but only to first order in $t/U$. Since $t/U$ is small in the bilayer samples studied so far, this assumption is not too restrictive. From now on, we shall work only to lowest non-trivial order in $t/U$, and
expand \( m_z \) as described in the Appendix. Thus we shall not include the effect of the rippling of \( m_z \) on the soliton width \( \xi \), since this would produce only small corrections (i.e., of higher order in \( t/U \)) to the results presented here. The soliton width \( \xi \) in Eq. (2.8) is therefore computed using \( m_z = m_{z,0} \), where \( m_{z,0} \) is the layer imbalance in the absence of interlayer tunneling, as defined in the Appendix. The value of \( m_{z,0} \) depends on the gate voltage \( V_g \), but not on the “rippling” effect (which is of higher order in \( t/U \)).

Because \( m_z = v_1 - v_2 \) is associated with differences in layer electron densities, the rippling has the effect of associating an electric-dipole density with each soliton. When the solitons are separated \( (L_z > \xi) \), the dipole-moment per unit length is (see the Appendix)

\[
\frac{\delta \rho}{\delta y} = \frac{e d \xi_0}{2 \pi \xi_0^2} \frac{m_{z,0}}{U} (1 - m_{z,0}^2)^{1/4} (4Q/Q_c - 3),
\]

which has a value of about \(-0.10e\) when \( m_{z,0} = 0.5 \) and \( Q = Q_c \). Because the solitons have associated dipole moments, the dominant interactions between solitons when the solitons are separated will be their dipole-dipole repulsion. In the limit that the solitons are well separated \( (L_z > \xi, d) \), the interaction per unit length between two solitons separated by a distance \( x \) is

\[
\frac{\mathcal{V}}{L} = \frac{(\delta \rho/\delta y)^2}{2\pi \epsilon s^2},
\]

so that the solitons repel each other with a force that falls off with distance as an inverse power (for \( v_1 \neq v_2 \)), rather than exponentially (for \( v_1 = v_2 \)). Summing all the dipole-dipole soliton interactions gives, in the thermodynamic limit,

\[
\frac{\mathcal{V}}{L_z L_y} = \frac{\pi}{12} \frac{(\delta \rho/\delta y)^2}{e L_y^3} = \frac{1}{96\pi^2} \frac{(\delta \rho/\delta y)^2}{e^3} Q_y^3,
\]

which is proportional to \((t/U)^2\) and is therefore small in magnitude. However, near the CI transition, the solitons are well separated and dipole interactions dominate the repulsions between the solitons, which has a strong effect on the compressional stiffness \( K_1 \) of the SL. The relation between the wave vector \( Q \) and the parameter \( \eta \) is obtained by minimizing the total energy per unit area with respect to \( Q \), at fixed \( Q \) (Ref. 16); when the layers are balanced, Eq. (2.10) results, and Eqs. (2.9) and (2.10) may be combined to obtain \( Q_s \) as a function of \( Q \). When the layers are imbalanced, Eq. (2.10) acquires an additional term due to the dipole interactions between solitons,

\[
\frac{Q/Q_c = E(\eta)/\eta + \frac{1}{\rho_s} \frac{\partial}{\partial Q_s} \frac{\mathcal{V}}{L_z L_y} = E(\eta)/\eta + C(Q_s/Q_c)^2,}
\]

where \( C \sim 0.14 \) for the hypothetical “typical” sample with \( m_{z,0} = 0.5 \) and \( Q = Q_c \). Near the CI transition Eq. (2.16) gives

\[
Q_s/Q_c \approx \sqrt{(Q/Q_c - 1)/C},
\]

\[
K_1/\rho_s \approx 2\sqrt{C(Q/Q_c - 1)}. \tag{2.17}
\]

The corresponding formula for the balanced \((v_1 = v_2)\) case are quite different: \( Q_s \sim 1/\ln(Q/Q_c) \) and \( K_1 \sim (Q/Q_c)^2 \), up to logarithmic corrections. Nonetheless, \( K_1 \) (and, by definition, \( Q_s \)) vanishes at the CI transition which has important consequences for the capacitance near the CI transition.

### D. Topological charge

We also note that when the layers are imbalanced, the soliton lines acquire a charge density when they are tilted or sheared. In the lowest Landau level, the pseudospin textures with topological charge possess an electric charge. The electric-charge density \( \delta \rho(\mathbf{r}) \) is just the topological-charge density (i.e., the Pontryagin index density \( 10 \)) times \(-ev_T\).

\[
\delta \rho = \frac{e v_T}{8\pi} \epsilon_{ij} \mathbf{m} \cdot (\partial_i \mathbf{m} \times \partial_j \mathbf{m}),
\]

where \( \epsilon_{ij} \) is the totally antisymmetric tensor of rank 2, and we sum over \( i \) and \( j \) over the values 1 and 2 ( \( x \) and \( y \) ). The total filling factor is defined as \( v_T = v_1 + v_2 \). Expression (2.18) in terms of \( m_z \) and \( \theta \) using

\[
m_z = \sqrt{1 - m_z^2} \cos \theta, \quad m_y = \sqrt{1 - m_z^2} \sin \theta \tag{2.19}
\]

gives

\[
\delta \rho(\mathbf{r}) = \frac{e v_T}{4\pi} [\partial_i \theta m_j - \partial_j m_i \partial_i \theta]
\]

\[
= \frac{e v_T}{4\pi} [\partial_i \theta (Q) \partial_j m_z - \partial_j m_z \partial_i \theta], \tag{2.20}
\]

where we have taken \( \mathbf{Q} \) to be along the \( \mathbf{x} \) direction. If we rotate the soliton lines by taking

\[
\tilde{\theta}_0(x) \rightarrow \tilde{\theta}_0(\alpha x + \beta y), \tag{2.21}
\]

and also transform the rippled layer imbalance according to

\[
\delta m_{z,1}(x) \rightarrow \delta m_{z,1}(\alpha x + \beta y), \tag{2.22}
\]

then the associated charge density is
\[ \delta \rho = - \frac{e}{4\pi} Q \delta \zeta \left( \alpha x + \beta y \right), \]  
which associates a spatially varying charge density proportional to \( \alpha x + \beta y \) to the tilted soliton lines. The integrated charge of the soliton lines remains zero, so the shear stiffness may not be greatly affected.

### E. Anisotropic transport

Application of a sufficiently strong in-plane magnetic field produces soliton lines that are parallel to the in-plane magnetic field. When the layers are imbalanced, these soliton lines possess dipole-charge densities. The resulting electric fields associated with the dipole-charge densities will make the conductivity anisotropic in the incommensurate phase. Transport parallel to the soliton lines (along the direction of the in-plane magnetic field, say \( -\hat{y} \)) is expected to be easier than in the direction perpendicular (along \( \hat{x} \)) to the soliton lines: i.e., when the layers are imbalanced, we expect

\[ \rho_{xx} > \rho_{yy}, \quad Q < Q_c, \]  
\[ \rho_{xx} < \rho_{yy}, \quad Q > Q_c. \]  

Thus a disparity between the longitudinal resistivities parallel and perpendicular to the in-plane magnetic field indicates the presence of an incommensurate soliton-lattice phase in the imbalanced system.

We note that anisotropic transport in the quantum Hall regime has been found in a very different context, in very high-mobility single-layer samples at high half-integer filling factors. In this case, the anisotropic transport is taken to indicate the presence of a spontaneously striped anisotropic charge density of the quantum Hall ground state. Here, the stripes are not spontaneous (for \( t \ll U \)), but are produced and oriented by the in-plane magnetic field. Nonetheless, the stripes (dipolar soliton lines) found here should also produce anisotropic transport.

### III. INTERLAYER CAPACITANCE

Capacitance measurements offer the possibility of directly probing the ground-state properties of two-dimensional electron systems, such as the thermodynamic compressibility. Although the differential gate capacitance, which measures how the charge on a gate changes with respect to changes in the gate voltage, is slightly affected by the compressibility of the electron gas in the occupied layer that is nearest the gate, it is almost entirely dominated by the large gate-to-layer distance of the device. A far more sensitive measurement of the electronic compressibility is provided by the Eisenstein ratio \( R_E \), which is an interlayer capacitance,

\[ R_E = \frac{\delta E_{12}}{\delta \epsilon}, \]  

where \( E_{12} \) is the electric field that exists between the layers, and \( \epsilon \) is the electric field between the gate and the nearest layer. Classically, conduction electrons in the layers should completely screen the electric fields produced by the gates, so that \( E_{12} = 0 \) and \( R_E = 0 \). Indeed, this result is approached when the layers are sufficiently far apart (beyond several hundred \( \AA \)). But, due to their finite density of states, the effectively two-dimensional electron layers cannot completely screen the gate electric fields, so \( E_{12} \) is nonzero when the gates are unbalanced.

The Eisenstein ratio has been measured experimentally in double-layer electron gas (2DEG) systems, and used to determine the negative compressibility of the low-density 2DEG; it has also been calculated theoretically at zero magnetic field and in 2LQH systems. The calculation of the Eisenstein ratio is discussed in some detail in Refs. 35–37; here we shall briefly outline only the key steps.

It is convenient to separate the total energy per unit area of the 2LQH system into an electrostatic part (i.e., the integrated electrostatic energy density \( \epsilon E_{12}^2 d/2 \) between the layers) and a many-body part \( \langle \epsilon \rangle \), which would be the energy per area for a system with neutralizing charge backgrounds in each layer. The chemical potential relative to the bottom of quantum well \( i \) is given by

\[ \mu_i = \partial \langle \epsilon \rangle / \partial n_i, \]  

where \( n_i = \nu_i/2\pi l^2 \) is the areal number density in layer \( i \). \( R_E \) can be expressed in terms of the interlayer separation \( d \) and the effective electronic lengths \( s_{ij} \), defined as

\[ s_{ij} = \frac{e^2}{2 \partial^2 \epsilon / \partial n_i} \frac{2 \pi l^2 \langle \epsilon \rangle}{\epsilon^2 / 4 \pi l}, \]  

from which it follows that \( s_{ji} = s_{ij} \). In the absence of interlayer interactions (the case considered in Ref. 35), \( s_{12} = s_{21} = 0 \) and the length \( s_{ij} \) is inversely proportional to the electronic compressibility \( \kappa_i \) in layer \( i \) (Ref. 36):

\[ s_{ii} = \frac{e}{\epsilon s_{ij}^2 \kappa_i}. \]  

If the front-gate and back-gate voltages are varied simultaneously so that the total layer density is kept fixed \( (\nu_1 + \nu_2 = 1) \), then

\[ R_E = \frac{\delta E_{12}}{\delta \epsilon} = \frac{s_1 + s_2}{d + s_1 + s_2}, \]  

where

\[ s_1 = s_{11} - s_{12}, \quad s_2 = s_{22} - s_{21}. \]  

If only one of the gate voltages is kept fixed, then the numerator of Eq. (3.5) is equal to either \( s_1 \) or \( s_2 \), instead of their sum. At high densities, \( s_1 \) (and thus \( R_E \)) are positive, but at sufficiently low densities, they become negative, resulting in the negative values of \( R_E \) that have been measured experimentally.

The Eisenstein ratio has been calculated for a \( \nu_f = 1 \) 2LQH state, although without a parallel magnetic field. It was found that, although the lengths \( s_i \) were negative, the criterion for stability against abrupt interlayer charge transfer was\( 165325-5 \)

\[ d + s_1 + s_2 > 0, \]
is still satisfied. It follows from Eqs. (3.5) and (3.7) that when \( R_E \) diverges, it signals an interlayer charge-transfer instability.

In the next two subsections, we calculate the contribution \( R_{E1} \) of the PT Hamiltonian to the Eisenstein ratio. \( R_{E1} \) contains the dependence of \( R_E \) on the tunneling \( t \), pseudospin stiffness \( \rho_s \), parallel magnetic field \( Q \), and layer imbalance \( m_{z,0} \). It is convenient to separate \( R_{E1} \) into two parts: one that is formally divergent at \( Q = Q_\epsilon \) when the layers are unbalanced \( (m_z \neq 0) \), and one that does not diverge, but which nevertheless exhibits a nontrivial dependence on \( Q \) and \( m_{z,0} \).

### A. Divergent contribution

We shall first consider only the contributions to the electronic lengths \( s_i \) that become negative and divergent at the CI transition. We therefore focus on how changes in the layer imbalance \( m_z \) allow one to cross through the CI transition, due to the \( m_z \) dependence of the critical parallel magnetic field (or of \( Q_\epsilon \)), as discussed in Sec. II B and in Ref. 16. We therefore consider only the PT part of the the energy per area in Eq. (2.1), and then only that part of the PT energy that depends on \( m_z \) through \( Q/Q_\epsilon \). (There are other terms that contribute to \( s_i \), but they do not give rise to a divergence at the CI transition, at least for \( t/U \approx 1 \), which is the usual situation in most samples.) Because we are focusing on the effects of passing through the CI transition, rather than the effects of changing the total filling factor, we restrict our attention to the case of fixed total filling factor \( \nu_T \). With these restrictions, the effect of taking a derivative of a function of \( Q/Q_\epsilon \) with respect to the layer filling factor \( \nu_i \) produces the equivalence

\[
\frac{\partial}{\partial \nu_i} \sim - \frac{\partial m_z}{\partial \nu_i} \frac{\partial Q}{\partial m_z} \frac{Q}{Q_\epsilon} \frac{\partial}{\partial Q} = \frac{(-1)^i}{2} \frac{m_z}{1-m_z^2} \frac{Q}{Q_\epsilon} \frac{\partial}{\partial Q}.
\]

(3.8)

Combining Eqs. (3.3) and (3.8) with the magnetization calculations of Ref. 16 gives

\[
\frac{s_1 + s_2}{l} \sim - \pi \left( \frac{m_z}{1-m_z^2} \right)^2 (Ql)^2 \frac{\rho_s}{e^2/4\pi \epsilon_l} \frac{\partial Q}{\partial Q},
\]

(3.9)

where we have kept only the contribution that diverges at the CI transition. The divergence occurs because nonzero \( m_z \) allows small changes in the layer filling factors to tune the system through the CI transition. The divergent part of the electronic lengths \( s_i \) are thus proportional to the in-plane differential magnetic susceptibility that was calculated in Ref. 16, and which diverges at the CI transition.

Unfortunately, the divergence in the electronic lengths \( s_i \) at the CI transition is weak, proportional to \( 1/\sqrt{Q/Q_\epsilon} \), with a small prefactor. For the hypothetical sample considered in this paper and \( m_{z,0} = 0.5 \), Eqs. (2.17) and (3.9) give

\[
\frac{s_1 + s_2}{l} \sim - \frac{0.005}{\sqrt{Q/Q_\epsilon - 1}},
\]

(3.10)

which requires \( (Q/Q_\epsilon - 1) \sim 10^{-5} \) to give a divergent value of the Eisenstein ratio (i.e., \( d + s_1 + s_2 = 0 \)), at least for a tunneling-matrix element \( t_0 = 0.5 \) meV. The required nearness to the CI transition (i.e., the smallness of \( Q/Q_\epsilon - 1 \)) is inversely proportional to \( t_0 \), so making \( t_0 \) smaller might be of some help. However, it may be that in practice, sample disorder smears out the CI transition well before the divergence in \( R_E \) can be approached.

### B. Nondivergent contributions

We saw in Sec. III A that, although there is a contribution to the interlayer capacitance (the Eisenstein ratio \( R_E \)) which is formally divergent at the CI transition, it may be difficult in practice to tune close enough to the transition to observe the divergence. It may be that the nondivergent contribution of the PT contribution to \( R_E \) is more easily detected. We calculate this contribution below for small \( 2t_0/U \).

For simplicity, we focus on the case of fixed total filling factor. Then it follows from the definition of \( R_E \) in Eq. (3.1), from Gauss’ law, from the definition of \( V_\epsilon \) in Eq. (2.6), and from Eq. (2.12), that we may express \( R_E \) as

\[
R_E = 1 - \frac{\bar{D}}{U} \frac{\partial (m_z)}{\partial m_{z,0}} \approx R_{E0} + R_{E1},
\]

(3.11)

\[
R_{E0} = \bar{D} \frac{1}{U} d (m_z)/\partial m_{z,0},
\]

\[
R_{E1} = - \frac{\bar{D}}{U} \frac{2t_0}{\sqrt{1-m_{z,0}^2}} \left( \frac{\bar{D}}{U} \frac{1}{\sqrt{1-m_{z,0}^2}} \right)^2 \left( \bar{Q}/\sqrt{Q_\epsilon} \right)^2.
\]

(3.12)

The largest contribution to the Eisenstein ratio is \( R_{E0} \), and it was this quantity that was calculated in Ref. 36. For the hypothetical sample parameters in the text, \( R_{E0} = -0.9 \), independent of \( m_{z,0} \) and \( Q \). \( R_E = R_{E0} \) when the interlayer-tunneling amplitude \( t_0 \) is very small, or when the parallel field is large \( (Q \) substantially larger than \( Q_\epsilon \)).

\( R_{E1} \) contains all the dependence of \( R_E \) on the in-plane magnetic field and layer imbalance. If \( t/U \) is made as large as possible, then by measuring the \( Q \) and \( m_{z,0} \) dependence of \( R_E \), it may be possible to measure \( R_{E1} \). In the commensurate phase \( (Q < Q_\epsilon) \), \( \bar{D} = 0 \), so

\[
R_{E1} = - \frac{\bar{D}}{U} \frac{2t_0}{\sqrt{1-m_{z,0}^2}} \left( \frac{1}{(1-m_{z,0}^2)} \right)^2 \frac{4}{\pi} \frac{Q}{Q_\epsilon^2}.
\]

(3.13)

In the absence of an in-plane magnetic field \( (Q = 0) \),

\[
R_{E1}(Q = 0) = - \frac{\bar{D}}{U} \frac{2t_0}{\sqrt{1-m_{z,0}^2}} \approx -0.39,
\]

(3.13)

for the hypothetical sample parameters described in the text, with \( m_{z,0} = 0.5 \), so that in this case \( R_{E1} \) is almost half the size of \( R_{E0} \). As \( Q \) increases, \( R_{E1} \) decreases in magnitude and,
provided that \(m_{z0} < \sqrt{1 - (\pi/4)^2} = 0.62\), passes through zero and becomes positive near the CI transition.

At the CI transition, this nondivergent part of \(R_{E1}\) is

\[
R_{E1}(Q = Q_c) = \frac{D_1}{U} \frac{2t_0/U}{\sqrt{1 - m_{z0}}} \left[ \frac{4}{\pi} - \frac{1}{1 - m_{z0}^2} \right].
\]

(3.14)

which for the sample parameters described in the text has a value of \(R_{E1}(Q = Q_c) = 0.16\) for \(m_{z0} = 0\) and \(R_{E1}(Q = Q_c) = 0.08\) for \(m_{z0} = 0.5\). In the incommensurate phase, \(R_{E1}\) drops rapidly to zero due to rapid spatial variations in \(\partial_\theta(x)\). By measuring how the in-plane magnetic field and layer imbalance affect the interlayer capacitance \(R_E\), it may be possible to estimate the pseudospin stiffness \(\rho_s\) in the commensurate phase, and also to detect the incommensurate phase, as signaled by a rapid decrease in the sensitivity of \(R_E\) to \(Q\) and \(M_{z0}\).

### IV. CONCLUSIONS

It has been shown that the incommensurate phase of a bilayer quantum Hall state has a “rippled” dipole charge density whenever the layers are unbalanced. The rippling arises because the layer imbalance \(m_z\) and the interlayer phase \(\theta\) are coupled through the \(m_z\) dependence of the effective tunneling energy \(t\) and pseudospin stiffness \(\rho_s\) in the Pokrovsky-Talapov part of the total energy in Eq. (2.1). This coupling between the layer imbalance and interlayer phase produces the rippled state when the translational symmetry of the phase is broken in the incommensurate phase. The rippled layer imbalance was calculated within the Hartree-Fock-gradient approximation and is illustrated in Fig. 2. The details of the calculation are given in the Appendix. We focused on the limit where the interlayer-tunneling energy is smaller than the charging energy of the bilayer \((t \ll U)\), which is the case for all bilayer samples which have been studied experimentally so far.

Because solitons have an associated dipole-moment per unit length (which we estimated in the Appendix) in the rippled state, well-separated soliton lines experience a power-law (inverse cube) repulsive-force per unit length, rather than the much weaker exponentially-decaying repulsion between solitons found in the balanced case. This has a strong effect on how the density of solitons depends on the in-plane magnetic field near the CI transition, as illustrated in Fig. 1.

The fact that solitons have an electric dipole moment in the rippled state makes it likely that they produce anisotropic transport. Transport parallel to the soliton lines \((\pm \hat{y})\) is likely to be easier than transport perpendicular to the soliton lines \((\pm \hat{x})\). We expect the ratio \(\rho_{xx}/\rho_{xy}\) to increase when the rippled state is entered. This could provide an experimental signature for the incommensurate state.

We calculated the interlayer capacitance, specifically the Eisenstein ratio \(R_E\), which is a sensitive measure of the electronic compressibility and of interlayer electronic correlations. When the layers are unbalanced, there is a contribution to \(R_E\) which diverges at the CI transition. However, observing this contribution is likely to be problematic because it requires that the CI transition be very sharp (unsmearable by disorder). We also calculated a contribution to \(R_E\) which does not diverge at the CI transition, but which nevertheless offers a good possibility of experimental detection. We calculated the in-plane magnetic-field dependence of \(R_E\), along with its dependence on layer imbalance, especially in the commensurate \((Q < Q_c)\) phase, up to the CI transition \((Q = Q_c)\), and discussed its behavior (rapid decline) in the incommensurate \((Q > Q_c)\) phase. By measuring \(R_E\), the pseudospin stiffness could be estimated and the incommensurate phase detected.

The existence of “rippled” layer densities in the incommensurate phase of unbalanced 2LQH systems is expected to be valid beyond the HFGA, although the size of the density variations will be reduced by quantum fluctuations. As long as the layers are “rippled,” they will make an anomalous contribution to the capacitance. Although fluctuation effects change the sizes of \(\rho_s\) and \(t\), the basic physics of producing “rippled” layer densities still holds beyond the HFA. In particular, \(\rho_s\) and \(t\) will still change with the layer imbalance \(m_z\), and this dependence on \(m_z\) will produce a coupling between \(m_z\) and the pseudospin angle \(\theta\), leading to nonuniform \(m_z\) in the incommensurate phase when \(\nu_1 \neq \nu_2\).

Several remarks about the observability of the capacitive effects found here are in order, especially the effects of including finite temperature and disorder. In practice, both finite temperature and disorder limit the minimum effective value of \((Q/Q_c - 1)\) that can be obtained. These are important topics which deserve further study; only some preliminary considerations are discussed here.

Ignoring the effects of disorder for the moment, it is important to note that the soliton lattice exists only at sufficiently low temperatures. The soliton lattice supports a finite-temperature Kosterlitz-Thouless (KT) transition due to dislocation-mediated melting of the lattice of soliton lines. The KT temperature for melting the soliton lattice is roughly \(k_B T_{KT} \approx (\pi/2) \sqrt{K_1 K_2}\), where \(K_1\) is the longitudinal stiffness and \(K_2\) is the transverse stiffness of the soliton lattice. Because \(K_1 \rightarrow 0\) as \(Q \rightarrow Q_c\), the KT temperature drops as the CI transition is approached. This sets a limit to how close one can get to the CI transition, which may be estimated in the case of finite layer imbalance by using \(K_1 = 2 \sqrt{c(Q/Q_c - 1)}\) [see Eq. (2.17)] and \(K_2 = \rho_s\). From these considerations, the requirement that \(T < T_{KT}\) gives

\[
(Q/Q_c - 1)^{1/4} > 2.4 T_{KT}^{1/4} \approx 2.4 \frac{T}{K} \quad (4.1)
\]

for the sample parameters used in the text. For \(T = 100\) mK, this yields \((Q/Q_c - 1) > 2.4 \times 10^{-4}\).

The smearing of the CI transition due to disorder is more problematic. Even in capacitance experiments designed to measure the (in)compressibility of the fractional quantum Hall (FQH) state (which would, in the absence of disorder, lead to a divergent \(R_E\) at odd-denominator filling factors), only finite changes in \(R_E\) are found at the FQH filling factors, due to the effects of disorder. It is to be expected that
disorder may eliminate any abrupt features in the interlayer capacitance $R_F$ in this case also. Strictly speaking, the long-range order of the SL is destroyed by any finite amount of disorder; presumably the SL has only a finite correlation length $\xi < \infty$ due to disorder. Roughly speaking, the maximum spacing between solitons ($L_s$) will be limited to $L_s < \xi_d$, so that

$$\frac{(Q/Q_c - 1)}{C} = \frac{\pi^2 \xi^2}{2 \xi_d^d}. \quad (4.2)$$

In practice, such disorder could arise from small variations in the local tunneling amplitudes $t$ or the spin stiffness $\rho_s$ due to minute variations in the interlayer barrier thickness and/or the layer separation. More theoretical work needs to be done to determine the limits imposed by disorder, but the issue of the observability of the CI transition in capacitance measurements must be settled experimentally.

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**APPENDIX: DIPOLE MOMENT**

In this Appendix, we estimate the dipole-moment per unit length associated with solitons in the incommensurate phase of an imbalanced 2LQH system. Expanding Eq. (2.1) in powers of ($t/U$) gives

$$m_z(r) = m_{z0} + m_{z1}(r), \quad (A1)$$

where

$$m_{z0} = V_z / U \quad (A2)$$

is the $t=0$ result for the layer imbalance, and

$$m_{z1}(r) = \frac{2t}{U} \frac{m_{z0}}{1 - m_{z0}^2} \left[ \xi^2 (\nabla \vec{\theta}_0 - Q)^2 - \cos \vec{\theta}_0 \right], \quad (A3)$$

to first order in $t/U$. Here $\vec{\theta}_0(r)$ is the soliton-line solution to the PT model for the values of $t$ and $\rho_s$ corresponding to setting $m_z = m_{z0}$ in Eq. (2.2).

For a single soliton, the lowest-order solution for $\vec{\theta}$ is

$$\vec{\theta}_0(x) = 4 \arctan \exp(x/\xi), \quad (A4)$$

and we associate an dipole-moment per unit length

![FIG. 3. Spatial dependence of the layer imbalance $\delta m_{z1}(x)$ associated with a single soliton, vs position for the hypothetical "typical" sample parameters described in the text.](Image)

where we have taken $Q$ to lie along the $\hat{x}$ direction. This gives

$$1 - \cos \vec{\theta}_0(x) = \frac{2}{\cosh^2(x/\xi)},$$

$$\xi \partial_x \vec{\theta}_0(x) = \frac{2}{\cosh(x/\xi)}. \quad (A5)$$

It is convenient to express $m_{z1}(r)$ as

$$m_{z1}(r) = m_{z0} + \delta m_{z1}(x), \quad (A6)$$

where

$$m_{z0} = \frac{2t}{U} \frac{m_{z0}}{1 - m_{z0}^2} \left( Q^2 \xi^2 - 1 \right) \quad (A7)$$

is the value of $m_{z0}$ in the commensurate ($\vec{\theta}_0 = 0$) phase. For the hypothetical sample described in the text, $m_{z1} \approx 0.048$ for $m_{z0} = 0.5$ and $Q = Q_c$. We associate the spatially-dependent part of $m_z$ (see Fig. 3) with the soliton line:

$$\delta m_{z1}(x) = \frac{2t}{U} \frac{m_{z0}}{1 - m_{z0}^2} \left[ \frac{4Q \xi}{\cosh(x/\xi)} - \frac{6}{\cosh^2(x/\xi)} \right]. \quad (A8)$$

The areal number density of layer $j$ is given by $n_j = \nu_j/(2\pi l^2)$. Therefore, the dipole-moment per unit area is

$$\frac{p}{L_x L_y} = - \frac{ed}{2\pi l^2} m_z, \quad (A9)$$

and we associate an dipole-moment per unit length...
\[
\frac{\delta p}{\delta y} = -\frac{ed}{2\pi l^2} \int_{-\infty}^{\infty} \delta m_{z}(x) \, dx
\]
\[
= -\frac{ed\xi}{2\pi l^2} \frac{8t}{U} \frac{m_{\infty}}{(1-m_{\infty}^2)^2} (\pi Q_0 - 3)
\]
\[
= -\frac{ed\xi_0}{2\pi l^2} \frac{2t_0}{U} \frac{m_{\infty}}{(1-m_{\infty}^2)^{1/4} (4Q/Q_c - 3)}.
\] (A10)

For the “typical” sample described in Sec. II A at layer imbalance \(m_{\infty} = 1/2\) and \(Q = Q_c\),
\[
\frac{\delta p}{\delta y} \approx -0.10e,
\] (A11)

where \(-e\) is the electric charge of an electron.

When the soliton lines do not overlap \((Q \text{ sufficiently near } Q_c)\), then \(\theta_0(x)\) is very nearly a periodic superposition of single-soliton solutions, spaced apart by \(L_s\):
\[
\theta_0(r) \approx 4 \sum_j \arctan \exp \left( (x - jL_s)/\xi \right).
\] (A12)

The dipolar interaction-energy per unit length between two parallel soliton lines separated by a distance \(x\) is
\[
\frac{\mathcal{V}_2(x)}{L_y} = \frac{(\delta p/\delta y)^2}{2\pi e \xi^2}.
\] (A13)

Thus the total dipole-interaction energy per unit area is
\[
\mathcal{V} = \frac{N_s}{L_s L_y} \sum_{j=1}^{\infty} \mathcal{V}_2(jL_y) = \frac{\pi}{12} \left( \frac{\delta p/\delta y}{} \right)^2 \frac{e}{\epsilon L_s^3}.
\]

where we have used the fact that the number of solitons is \(N_s = L_s/L_y\).

The relation between the wave vector \(Q\) and the parameter \(\eta\) is obtained by minimizing the total energy per unit area with respect to \(Q\), at fixed \(Q_c\) when the layers are balanced, Eq. (2.10) results, and Eqs. (2.9) and (2.10) may be combined to obtain \(Q_s\) as a function of \(Q\). When the layers are imbalanced, Eq. (2.10) acquires an additional term due to the dipole interactions between solitons,
\[
Q/Q_c = E(\eta)/\eta + \frac{1}{\rho_s Q_c} \frac{\partial}{\partial Q_c} \frac{\mathcal{V}}{L_s L_y} = E(\eta)/\eta + C(Q_s/Q_c)^2,
\] (A15)

where
\[
C = \frac{1}{8\pi} \frac{(\delta p/\delta y)^2}{e} \frac{\epsilon^2}{\rho_s} \frac{1}{Q_c L_s h}.
\] (A16)

and \(C \sim 0.14\) for the hypothetical “typical” sample with \(m_{\infty} = 0.5\) and \(Q = Q_c\). Because the interaction between separated solitons is an inverse-power law (when unbalanced) rather than exponentially decaying function (when balanced), \(Q_s\) and \(K_1\) are proportional to \(\sqrt{Q - Q_c}\) near the CI transition.

2 The Quantum Hall Effect, edited by R. E. Prange and S. M. Girvin (Springer-Verlag, New York, 1990), and references therein.
38 Thermal fluctuations will renormalize the SL stiffness and lead to other interesting effects; for recent work, see: E. Papa and A. Tsvelik, cond-mat/0201343 (unpublished); S. Park, K. Moon, C. Ahn, J. Yeo, C. Rim, and B.H. Lee, cond-mat/0203498 (unpublished).
39 M.P.A. Fisher (private communication).